MATH 511 HOMEWORK 3
DUE THURSDAY, JULY 4, 2019

3. Orthogonality

3.1. Orthogonal Vectors and Subspaces.

Exercise 1. Find the lengths and the inner product of $x = (1, 4, 0, 2)$ and $y = (2, -2, 1, 3)$.

Solution: Write

$$\begin{aligned} \|x\| &= \sqrt{1^2 + 4^2 + 0^2 + 2^2} \\ &= \sqrt{1 + 16 + 0 + 4} \\ &= \sqrt{21} \\ \|y\| &= \sqrt{2^2 + (-2)^2 + 1^2 + 3^2} \\ &= \sqrt{4 + 4 + 1 + 9} \\ &= \sqrt{18} \end{aligned}$$

Hence, $\|x\| = \sqrt{21}$ and $\|y\| = \sqrt{18}$.

Next,

$$\begin{aligned} x \cdot y &= 1(2) + 4(-2) + 0(1) + 2(3) \\ &= 2 - 8 + 0 + 6 \\ &= 0 \end{aligned}$$

We conclude that x and y are orthogonal.

Exercise 5. Which pairs are orthogonal among the vectors v_1, v_2, v_3, v_4?

$$v_1 = \begin{bmatrix} 1 \\ 2 \\ -2 \\ 1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 4 \\ 0 \\ 4 \\ 0 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 \\ -1 \\ -1 \\ -1 \end{bmatrix}, \quad v_4 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

Solution: Consider

$$\begin{aligned} v_1 \cdot v_2 &= 1(4) + 2(0) + (-2)(4) + 1(0) \\ &= 4 + 0 - 8 + 0 \\ &= -4 \\ v_1 \cdot v_3 &= 1(1) + 2(-1) + (-2)(-1) + 1(-1) \\ &= 1 - 2 + 2 - 1 \end{aligned}$$

$$\begin{aligned}
&= 0 \\
v_1 \cdot v_4 &= 1(1) + 2(1) + (-2)(1) + 1(1) \\
&= 1 + 2 - 2 + 1 \\
&= 2 \\
v_2 \cdot v_3 &= 4(1) + 0(-1) + 4(-1) + 0(-1) \\
&= 4 + 0 - 4 + 0 \\
&= 0 \\
v_2 \cdot v_4 &= 4(1) + 0(1) + 4(1) + 0(1) \\
&= 4 + 0 + 4 + 0 \\
&= 8 \\
v_3 \cdot v_4 &= 1(1) + (-1)(1) + (-1)(1) + (-1)(1) \\
&= 1 - 1 - 1 - 1 \\
&= -2
\end{aligned}$$

From our calculations, we conclude that v_1 and v_3 are orthogonal and v_2 and v_3 are orthogonal.

Exercise 12. Find a basis for the orthogonal complement of the row space of A:

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 4 \end{bmatrix}.$$

Split $x = (3, 3, 3)$ into a row space component x_r and a null space component x_n.

Solution: By 3D on p. 146, we see that the row space of A is orthogonal complement of the null space of A, which is to say

$$(\mathbf{C}(A^T))^{\perp} = \mathbf{N}(A).$$

Hence, we find a basis for the null space of A.

We begin by reducing A:

$$\begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 4 \end{bmatrix} \xrightarrow{-R_1+R_2 \to R_2} \begin{bmatrix} \mathbf{1} & \mathbf{0} & 2 \\ \mathbf{0} & \mathbf{1} & 2 \end{bmatrix}.$$

Of (a, b, c), c is the free variable. To compute the null space, we write

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} a + 2c \\ b + 2c \end{bmatrix}.$$

This implies $a = -2c$ and $b = -2c$ and so a basis for the null space (the orthogonal complement of the row space of A) is

$$\left\{ \begin{bmatrix} -2 \\ -2 \\ 1 \end{bmatrix} \right\}.$$

As for the row space of A, we consider that the reduced U is

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \end{bmatrix}.$$

Hence, a basis for the row space is

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \right\}.$$

Next, we decompose $x = (3, 3, 3)$ into the row space and null space. We look at two methods of approaching this.

Method 1: Determine the row space of A

Observe that

$$x = \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix} = a \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + b \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} + c \begin{bmatrix} -2 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} a - 2c \\ b - 2c \\ 2a + 2b + c \end{bmatrix}$$

and see that

$$3 = a - 2c \;\Rightarrow\; c = \frac{a-3}{2}$$
$$3 = b - 2c \;\Rightarrow\; c = \frac{b-3}{2}$$

which implies $a = b$. Then, we see that

$$\begin{aligned} & 3 = 2a + 2b + c \\ & \quad = 2a + 2a + \frac{a-3}{2} \\ \Rightarrow \quad & 6 = 8a + a - 3 \\ & \quad = 9a - 3 \\ \Rightarrow \quad & 9 = 9a \\ \Rightarrow \quad & a = 1 \end{aligned}$$

Thus, we see that $a = 1 = b$ and $c = \dfrac{1-3}{2} = -1$. From our work, we see

$$x_r = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}$$

and

$$x_n = -\begin{bmatrix} -2 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}.$$

We conclude

$$x = \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}}_{x_r} + \underbrace{\begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}}_{x_n}.$$

Method 2: Using the dot product

Since $x = x_r + cx_n$, then $x_r = x - cx_n$. We need $x_r \cdot x_n = 0$, which means we must choose a c such that $(x - cx_n) \cdot x_n = 0$. Write

$$\begin{aligned} 0 &= (x - cx_n) \cdot x_n \\ &= \left(\begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix} - c \begin{bmatrix} -2 \\ -2 \\ 1 \end{bmatrix} \right) \cdot \begin{bmatrix} -2 \\ -2 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 3+2c \\ 3+2c \\ 3-c \end{bmatrix} \cdot \begin{bmatrix} -2 \\ -2 \\ 1 \end{bmatrix} \\ &= -2(3+2c) - 2(3+2c) + (3-c) \\ &= -6 - 4c - 6 - 4c + 3 - c \\ &= -9 - 9c \\ \Rightarrow \quad 9c &= -9 \\ \Rightarrow \quad c &= -1 \end{aligned}$$

Now, from this, we see that

$$x_r = x + x_n = \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix} + \begin{bmatrix} -2 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}.$$

We conclude

$$\begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}}_{x_r} + \underbrace{\begin{bmatrix} -2 \\ -2 \\ 1 \end{bmatrix}}_{x_n}.$$

Note 1. We need to be careful when computing x_r. Simply because $x - cx_n$ is a particular solution does not mean that it is in the row space. We need to check that $x - cx_n$ is actually orthogonal to the null space as well.

3.2. **Cosines and Projections onto Lines.**

Exercise 3. What multiple of $a = (1, 1, 1)$ is closest to the point $b = (2, 4, 4)$? Find also the point closest to a on the line through b.

Solution: The formula for the projection of b onto a is

$$\text{proj}_a b = \frac{a^T b}{a^T a}\, a.$$

Hence,

$$\begin{aligned}
\text{proj}_a b &= \frac{\begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \\ 4 \end{bmatrix}}{\begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \\
&= \frac{1(2) + 1(4) + 1(4)}{1(1) + 1(1) + 1(1)} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \\
&= \frac{2 + 4 + 4}{1 + 1 + 1} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \\
&= \frac{10}{3} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}
\end{aligned}$$

Next, we compute the projection of a onto b. We write

$$\begin{aligned}
\text{proj}_b a &= \frac{b^T a}{b^T b}\, b \\
&= \frac{\begin{bmatrix} 2 & 4 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}{\begin{bmatrix} 2 & 4 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \\ 4 \end{bmatrix}} \begin{bmatrix} 2 \\ 4 \\ 4 \end{bmatrix} \\
&= \frac{2(1) + 4(1) + 4(1)}{2(2) + 4(4) + 4(4)} \begin{bmatrix} 2 \\ 4 \\ 4 \end{bmatrix}
\end{aligned}$$

$$= \frac{2+4+4}{4+16+16} \begin{bmatrix} 2 \\ 4 \\ 4 \end{bmatrix}$$

$$= \frac{10}{36} \begin{bmatrix} 2 \\ 4 \\ 4 \end{bmatrix}$$

$$= \frac{5}{9} \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}.$$

Note 2. This formula only applies to the projection onto a line. Any other type of projection needs to use the formula $P = A(A^TA)^{-1}A^Tb$.

Exercise 11.

(a) Find the projection matrix P_1 onto the line through $a = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ and also the matrix P_2 that projects onto the line perpendicular to a.

Solution: Write

$$P_1 = \left(\frac{1}{a^Ta}\right) aa^T$$

$$= \frac{1}{\begin{bmatrix} 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix}} \begin{bmatrix} 1 \\ 3 \end{bmatrix} \begin{bmatrix} 1 & 3 \end{bmatrix}$$

$$= \frac{1}{1(1)+3(3)} \begin{bmatrix} 1(1) & 1(3) \\ 3(1) & 3(3) \end{bmatrix}$$

$$= \frac{1}{1+9} \begin{bmatrix} 1 & 3 \\ 3 & 9 \end{bmatrix}$$

$$= \frac{1}{10} \begin{bmatrix} 1 & 3 \\ 3 & 9 \end{bmatrix}$$

To find a $b = (b_1, b_2)$ perpendicular to a, consider that $a \cdot b = 0$ implies $a \perp b$. Write

$$0 = a \cdot b = 1(b_1) + 3(b_2) = b_1 + 3b_2 \;\Rightarrow\; b_1 = -3b_2.$$

Take $b = (-3, 1)$.

Now, we write

$$\begin{aligned}
P_2 &= \left(\frac{1}{b^T b}\right) bb^T \\
&= \frac{1}{\begin{bmatrix} -3 & 1 \end{bmatrix}\begin{bmatrix} -3 \\ 1 \end{bmatrix}} \begin{bmatrix} -3 \\ 1 \end{bmatrix}\begin{bmatrix} -3 & 1 \end{bmatrix} \\
&= \frac{1}{(-3)(-3)+1(1)} \begin{bmatrix} -3(-3) & -3(1) \\ 1(-3) & 1(1) \end{bmatrix} \\
&= \frac{1}{9+1} \begin{bmatrix} 9 & -3 \\ -3 & 1 \end{bmatrix} \\
&= \frac{1}{10} \begin{bmatrix} 9 & -3 \\ -3 & 1 \end{bmatrix}
\end{aligned}$$

Note 3. By construction, $I - P$ is the projection onto the line perpendicular to a. This method would have been easier than what I did but I had already typed it up so there you go, extra verification.

(b) Compute $P_1 + P_2$ and $P_1 P_2$ and explain.

Solution: We expect that $P_1 + P_2 = I$ and $P_1 P_2 = 0$ because P_1, P_2 are orthogonal projections onto spaces which are orthogonal complements.

Note 4. This is what I believe the question is asking to "explain," but I honestly don't know. The "explanation" at the end of the book just states what we computed in words, but I don't understand how that's an explanation.

We show that what we expect is indeed the case. Write

$$\begin{aligned}
P_1 + P_2 &= \frac{1}{10}\begin{bmatrix} 1 & 3 \\ 3 & 9 \end{bmatrix} + \frac{1}{10}\begin{bmatrix} 9 & -3 \\ -3 & 1 \end{bmatrix} \\
&= \frac{1}{10}\left(\begin{bmatrix} 1 & 3 \\ 3 & 9 \end{bmatrix} + \begin{bmatrix} 9 & -3 \\ -3 & 1 \end{bmatrix}\right) \\
&= \frac{1}{10}\left(\begin{bmatrix} 1+9 & 3-3 \\ 3-3 & 9+1 \end{bmatrix}\right) \\
&= \frac{1}{10}\begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.
\end{aligned}$$

Next,

$$\begin{aligned} P_1P_2 &= \left(\frac{1}{10}\begin{bmatrix} 1 & 3 \\ 3 & 9 \end{bmatrix}\right)\left(\frac{1}{10}\begin{bmatrix} 9 & -3 \\ -3 & 1 \end{bmatrix}\right) \\ &= \frac{1}{100}\left(\begin{bmatrix} 1 & 3 \\ 3 & 9 \end{bmatrix}\begin{bmatrix} 9 & -3 \\ -3 & 1 \end{bmatrix}\right) \\ &= \frac{1}{100}\begin{bmatrix} 1(9)+3(-3) & 1(-3)+3(1) \\ 3(9)+9(-3) & 3(-3)+9(1) \end{bmatrix} \\ &= \frac{1}{100}\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \end{aligned}$$

Exercise 12. Find the matrix that projects every point in the plane onto the line $x + 2y = 0$.

Solution: First, we observe that the solutions to $x + 2y = 0$ are multiples of the vector $a = (-2, 1)$ and so we find $P = \dfrac{aa^T}{a^Ta}$. Second, we compute this matrix P. Write

$$\begin{aligned} P &= \left(\frac{1}{a^Ta}\right)aa^T \\ &= \frac{1}{\begin{bmatrix} -2 & 1 \end{bmatrix}\begin{bmatrix} -2 \\ 1 \end{bmatrix}}\begin{bmatrix} -2 \\ 1 \end{bmatrix}\begin{bmatrix} -2 & 1 \end{bmatrix} \\ &= \frac{1}{(-2)(-2)+1(1)}\begin{bmatrix} -2(-2) & -2(1) \\ 1(-2) & 1(1) \end{bmatrix} \\ &= \frac{1}{4+1}\begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix} \\ &= \frac{1}{5}\begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix} \end{aligned}$$

3.3. **Projections and Least Squares.**

Exercise 3. Solve $Ax = b$ by least squares, find $p = A\widehat{x}$ if

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$$

Verify that the error $b - p$ is perpendicular to the columns for A.

Solution: We see that

$$A^T = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

and so

$$\begin{aligned} A^T A &= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1(1)+0(0)+1(1) & 1(0)+0(1)+1(1) \\ 0(1)+1(0)+1(1) & 0(0)+1(1)+1(1) \end{bmatrix} \\ &= \begin{bmatrix} 1+0+1 & 0+0+1 \\ 0+0+1 & 0+1+1 \end{bmatrix} \\ &= \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \end{aligned}$$

Note 5 (Useful Fact). If $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is an invertible matrix, then

$$M^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

From this note, we see that

$$\begin{aligned} (A^T A)^{-1} &= \frac{1}{2(2)-1(1)} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \\ &\frac{1}{4-1} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \\ &= \frac{1}{3} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \end{aligned}$$

Next,

$$\begin{aligned} \widehat{x} &= (A^T A)^{-1} A^T b \\ &= \frac{1}{3} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \\ &= \frac{1}{3} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 1(1)+0(1)+1(0) \\ 0(1)+1(1)+1(0) \end{bmatrix} \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{3}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 1+0+0 \\ 0+1+0 \end{bmatrix} \\
&= \frac{1}{3}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} \\
&= \frac{1}{3}\begin{bmatrix} 2(1)+(-1)(1) \\ -1(1)+2(1) \end{bmatrix} \\
&= \frac{1}{3}\begin{bmatrix} 2-1 \\ -1+2 \end{bmatrix} \\
&= \frac{1}{3}\begin{bmatrix} 1 \\ 1 \end{bmatrix}
\end{aligned}
$$

Now, we can find our projection. We have

$$
\begin{aligned}
p = A\widehat{x} &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}\left(\frac{1}{3}\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) \\
&= \frac{1}{3}\begin{bmatrix} 1(1)+0(1) \\ 0(1)+1(1) \\ 1(1)+1(1) \end{bmatrix} \\
&= \frac{1}{3}\begin{bmatrix} 1+0 \\ 0+1 \\ 1+1 \end{bmatrix} \\
&= \frac{1}{3}\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}
\end{aligned}
$$

From this, we see that

$$
\begin{aligned}
b - p &= \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - \frac{1}{3}\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \\
&= \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 1/3 \\ 1/3 \\ 2/3 \end{bmatrix}
\end{aligned}
$$

$$= \begin{bmatrix} 2/3 \\ 2/3 \\ -2/3 \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} 2 \\ 2 \\ -2 \end{bmatrix}$$

Finally, we show that $b-p$ is perpendicular to the columns of A by demonstrating that $b-p$ is in the **left null space of** A:

$$A^T(b-p) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}\left(\frac{1}{3}\begin{bmatrix} 2 \\ 2 \\ -2 \end{bmatrix}\right) = \frac{1}{3}\begin{bmatrix} 1(2)+0(2)+1(-2) \\ 0(2)+1(2)+1(-2) \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 2+0-2 \\ 0+2-2 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Exercise 4. Write out $E^2 = \|Ax-b\|^2$ and set to zero its derivatives with respect to u and v, if

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}, \quad x = \begin{bmatrix} u \\ v \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}.$$

Compare the resulting equations with $A^TA\widehat{x} = A^Tb$, confirming that calculus as well as geometry gives the normal equations. Find the solution $\widehat{x}$ and the projection $p = A\widehat{x}$. Why is $p = b$?

Solution: We begin with computing $Ax-b$:

$$Ax-b = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} - \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}$$

$$= \begin{bmatrix} 1(u)+0(v) \\ 0(u)+1(v) \\ 1(u)+1(v) \end{bmatrix} - \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}$$

$$= \begin{bmatrix} u \\ v \\ u+v \end{bmatrix} - \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}$$

$$= \begin{bmatrix} u-1 \\ v-3 \\ u+v-4 \end{bmatrix}$$

Then, we consider

$$\begin{aligned} E^2 = \|Ax-b\|^2 &= (u-1)^2 + (v-3)^2 + ((u+v)-4)^2 \\ &= (u^2-2u+1) + (v^2-6v+9) + ((u+v)^2 - 8(u+v) + 16) \\ &= (u^2-2u+1) + (v^2-6v+9) + (u^2+2uv+v^2-8u-8v+16) \\ &= 2u^2 + 2v^2 + 2uv - 10u - 14v + 26 \end{aligned}$$

Note 6. It would have been much easier to just keep E^2 as $(u-1)^2+(v-3)^2+((u+v)-4)^2$ but I made some odd choices for this homework.

Taking the partial derivatives and setting them equal to 0,

$$\left.\begin{array}{rcrcl} (E^2)_u &=& 4u+2v-10 &=& 0 \\ (E^2)_v &=& 4v+2u-14 &=& 0 \end{array}\right\} \Longrightarrow \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 10 \\ 14 \end{bmatrix}.$$

Next, we see from exercise # 3 above,

$$A^TA = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}.$$

Since

$$A^Tb = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 1(1)+0(3)+1(4) \\ 0(1)+1(3)+1(4) \end{bmatrix} = \begin{bmatrix} 1+0+4 \\ 0+3+4 \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \end{bmatrix},$$

we see that $A^TA\,\widehat{x} = A^Tb$ becomes

$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \end{bmatrix}.$$

We observe that this is the same as what we found from the calculus above.

The solution to this system of equations is given by

$$\left[\begin{array}{cc|c} 4 & 2 & 10 \\ 2 & 4 & 14 \end{array}\right] \overset{\frac{1}{2}R_1\to R_1}{\underset{-\frac{1}{2}R_2\to R_2}{\longrightarrow}} \left[\begin{array}{cc|c} 2 & 1 & 5 \\ 1 & 2 & 7 \end{array}\right] \overset{-\frac{1}{2}R_1+R_2\to R_2}{\longrightarrow} \left[\begin{array}{cc|c} 2 & 1 & 5 \\ 0 & 3/2 & 9/2 \end{array}\right]$$

$$\overset{\frac{2}{3}R_2\to R_2}{\longrightarrow} \left[\begin{array}{cc|c} 2 & 1 & 5 \\ 0 & 1 & 3 \end{array}\right] \overset{-R_2+R_1\to R_1}{\longrightarrow} \left[\begin{array}{cc|c} 2 & 0 & 2 \\ 0 & 1 & 3 \end{array}\right] \overset{\frac{1}{2}R_1\to R_1}{\longrightarrow} \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 3 \end{array}\right]$$

Hence,

$$\widehat{x} = \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}.$$

Next, we compute p. We begin by computing $(A^TA)^{-1}$:

$$(A^TA)^{-1} = \frac{1}{2(2)-1(1)}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}.$$

Then, we see

$$\begin{aligned} P = A(A^TA)^{-1}A^T &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}\left(\frac{1}{3}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\right)\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \\ &= \frac{1}{3}\begin{bmatrix} 2 & -1 \\ -1 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \\ &= \frac{1}{3}\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}. \end{aligned}$$

Moreover,

$$\begin{aligned} p = Pb &= \frac{1}{3}\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix} \\ &= \frac{1}{3}\begin{bmatrix} 2-3+4 \\ -1+6+4 \\ 1+3+8 \end{bmatrix} \\ &= \frac{1}{3}\begin{bmatrix} 3 \\ 9 \\ 12 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix} = b \end{aligned}$$

Because b is in the column space of A, we expect $p = b$. Indeed,

$$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + 3\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix} = b.$$

Exercise 6. Find the projection of b onto the column space of A:

$$A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ -2 & 4 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 2 \\ 7 \end{bmatrix}.$$

Split b into $p + q$, with p in the column space and q perpendicular to that space. Which of the four subspaces contains q?

Solution: The projection we are looking for is

$$P = A(A^TA)^{-1}A^T.$$

We first compute $(A^TA)^{-1}$. We have

$$A^T = \begin{bmatrix} 1 & 1 & -2 \\ 1 & -1 & 4 \end{bmatrix}$$

and so

$$\begin{aligned} A^TA &= \begin{bmatrix} 1 & 1 & -2 \\ 1 & -1 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ -2 & 4 \end{bmatrix} \\ &= \begin{bmatrix} 1+1+4 & 1-1-8 \\ 1-1-8 & 1+1+16 \end{bmatrix} \\ &= \begin{bmatrix} 6 & -8 \\ -8 & 18 \end{bmatrix} \end{aligned}$$

By note (5), we see that

$$\begin{aligned} (A^TA)^{-1} &= \frac{1}{6(18)-(-8)(-8)} \begin{bmatrix} 18 & 8 \\ 8 & 6 \end{bmatrix} \\ &= \frac{1}{44} \begin{bmatrix} 18 & 8 \\ 8 & 6 \end{bmatrix} \\ &= \frac{1}{22} \begin{bmatrix} 9 & 4 \\ 4 & 3 \end{bmatrix} \end{aligned}$$

Next, we compute $P = A(A^TA)^{-1}A^T$:

$$\begin{aligned} P &= \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ -2 & 4 \end{bmatrix} \left(\frac{1}{22} \begin{bmatrix} 9 & 4 \\ 4 & 3 \end{bmatrix} \right) \begin{bmatrix} 1 & 1 & -2 \\ 1 & -1 & 4 \end{bmatrix} \\ &= \frac{1}{2} \begin{bmatrix} 9+4 & 4+3 \\ 9-4 & 4-3 \\ -18+16 & -8+12 \end{bmatrix} \begin{bmatrix} 1 & 1 & -2 \\ 1 & -1 & 4 \end{bmatrix} \\ &= \frac{1}{22} \begin{bmatrix} 13 & 7 \\ 5 & 1 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 & -2 \\ 1 & -1 & 4 \end{bmatrix} \end{aligned}$$

$$= \frac{1}{22}\begin{bmatrix} 13+7 & 13-7 & -26+28 \\ 5+1 & 5-1 & -10+4 \\ -2+4 & -2-4 & 4+16 \end{bmatrix}$$

$$= \frac{1}{22}\begin{bmatrix} 20 & 6 & 2 \\ 6 & 4 & -6 \\ 2 & -6 & 20 \end{bmatrix}$$

$$= \frac{1}{11}\begin{bmatrix} 10 & 3 & 1 \\ 3 & 2 & -3 \\ 1 & -3 & 10 \end{bmatrix}$$

From this we see that the projection of b onto the column space of A is given by

$$p = Pb$$

$$= \frac{1}{11}\begin{bmatrix} 10 & 3 & 1 \\ 3 & 2 & -3 \\ 1 & -3 & 10 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 7 \end{bmatrix}$$

$$= \frac{1}{11}\begin{bmatrix} 10+6+7 \\ 3+4-21 \\ 2-6+70 \end{bmatrix}$$

$$= \frac{1}{11}\begin{bmatrix} 23 \\ -14 \\ 65 \end{bmatrix}$$

Now, because our projections are orthogonal, we can write

$$b = Pb + (I-P)b = p + q.$$

We have already computed p, so we need only compute q. Observe that

$$Q = I - P$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \frac{1}{11}\begin{bmatrix} 10 & 3 & 1 \\ 3 & 2 & -3 \\ 1 & -3 & 10 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 10/11 & 3/11 & 1/11 \\ 3/11 & 2/11 & -3/11 \\ 1/11 & -3/11 & 10/11 \end{bmatrix}$$

$$= \begin{bmatrix} 1-10/11 & -3/11 & -1/11 \\ -3/11 & 1-2/11 & 3/11 \\ -1/11 & 3/11 & 1-10/11 \end{bmatrix}$$

$$= \begin{bmatrix} 1/11 & -3/11 & -1/11 \\ -3/11 & 9/11 & 3/11 \\ -1/11 & 3/11 & 1/11 \end{bmatrix}$$

$$= \frac{1}{11} \begin{bmatrix} 1 & -3 & -1 \\ -3 & 9 & 3 \\ -1 & 3 & 1 \end{bmatrix}$$

Then we see that

$$q = Qb = \frac{1}{11} \begin{bmatrix} 1 & -3 & -1 \\ -3 & 9 & 3 \\ -1 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 7 \end{bmatrix}$$

$$= \frac{1}{11} \begin{bmatrix} 1-6-7 \\ -3+18+21 \\ -1+6+7 \end{bmatrix}$$

$$= \frac{1}{11} \begin{bmatrix} -12 \\ 36 \\ 12 \end{bmatrix}$$

$$= \frac{12}{11} \begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix}$$

Hence, we see that

$$b = p + q$$

$$= \frac{1}{11} \begin{bmatrix} 23 \\ -14 \\ 65 \end{bmatrix} + \frac{1}{11} \begin{bmatrix} -12 \\ 36 \\ 12 \end{bmatrix}$$

$$= \frac{1}{11} \begin{bmatrix} 23-12 \\ -14+36 \\ 65+12 \end{bmatrix}$$

$$= \frac{1}{11} \begin{bmatrix} 11 \\ 22 \\ 77 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 2 \\ 7 \end{bmatrix}$$

Moreover, observe that

$$\begin{aligned} Pq &= \frac{1}{11}\begin{bmatrix} 10 & 3 & 1 \\ 3 & 2 & -3 \\ 1 & -3 & 10 \end{bmatrix}\left(\frac{12}{11}\begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix}\right) \\ &= \frac{12}{121}\begin{bmatrix} -10+9+1 \\ -3+6-3 \\ -1-9+10 \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \end{aligned}$$

Hence, we see that q is in the space perpendicular to the column space of A. Because

$$\begin{aligned} A^T q &= \begin{bmatrix} 1 & 1 & -2 \\ 1 & -1 & 4 \end{bmatrix}\left(\frac{12}{11}\begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix}\right) \\ &= \frac{12}{11}\begin{bmatrix} -1+3-2 \\ -1-3+4 \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \end{aligned}$$

we see that q is an element of $\mathbf{N}(A^T)$, the **left null space of** A. We should expect this since $(\mathbf{C}(A))^\perp = \mathbf{N}(A^T)$ by 3D p. 146.

Exercise 7. Find the projection matrix P onto the space spanned by $a_1 = (1, 0, 1)$ and $a_2 = (1, 1, -1)$.

Solution: Let

$$A = [a_1\, a_2] = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & -1 \end{bmatrix}.$$

We compute the projection matrix $P = A(A^TA)^{-1}A^T$ and we begin by finding $(A^TA)^{-1}$. Write

$$\begin{aligned} A^TA &= \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & -1 \end{bmatrix} \\ &= \begin{bmatrix} 1+0+1 & 1+0-1 \\ 1+0-1 & 1+1+1 \end{bmatrix} \end{aligned}$$

$$= \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$$

Since this is diagonal, we immediately conclude

$$(A^TA)^{-1} = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/3 \end{bmatrix}.$$

Next,

$$\begin{aligned} P &= A(A^TA)^{-1}A^T \\ &= \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1/2 & 0 \\ 0 & 1/3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & -1 \end{bmatrix} \\ &= \begin{bmatrix} 1/2 & 1/3 \\ 0 & 1/3 \\ 1/2 & -1/3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & -1 \end{bmatrix} \\ &= \begin{bmatrix} 1/2+1/3 & 1/3 & 1/2-1/3 \\ 1/3 & 1/3 & 1/2+1/3 \\ 1/2-1/3 & -1/3 & 1/2+1/3 \end{bmatrix} \\ &= \begin{bmatrix} 5/6 & 1/3 & 1/6 \\ 1/3 & 1/3 & -1/3 \\ 1/6 & -1/3 & 5/6 \end{bmatrix} \\ &= \frac{1}{6} \begin{bmatrix} 5 & 2 & 1 \\ 2 & 2 & -2 \\ 1 & -2 & 5 \end{bmatrix} \end{aligned}$$

Note 7. The answer at the back of the book is incorrect.

Exercise 24. Find the best straight-line fit to the following measurements, and sketch your solution:

$$\begin{array}{cccccc|cccccc} y & = & 2 & \text{at} & t & = & -1 & y & = & 0 & \text{at} & t & = & 0 \\ y & = & -3 & \text{at} & t & = & 1 & y & = & -5 & \text{at} & t & = & 2. \end{array}$$

Solution: Because we are finding a straight line fit, we want $y = \widehat{C} + \widehat{D}t$ where $\widehat{x} = (\widehat{C}, \widehat{D})$ is the best approximation to $x = (C, D)$ for

$$Ax = \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ -3 \\ -5 \end{bmatrix} = b.$$

To find $\widehat{x}$, we compute Qb where $Q = (A^TA)^{-1}A^T$. So

$$A^T = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 0 & 1 & 2 \end{bmatrix}$$

and

$$\begin{aligned} A^TA &= \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 1+1+1+1 & -1+0+1+2 \\ -1+0+1+2 & 1+0+1+4 \end{bmatrix} \\ &= \begin{bmatrix} 4 & 2 \\ 2 & 6 \end{bmatrix}. \end{aligned}$$

By note (5),

$$\begin{aligned} (A^TA)^{-1} &= \frac{1}{4(6)-2(2)} \begin{bmatrix} 6 & -2 \\ -2 & 4 \end{bmatrix} \\ &= \frac{1}{20} \begin{bmatrix} 6 & -2 \\ -2 & 4 \end{bmatrix} \\ &= \frac{1}{10} \begin{bmatrix} 3 & -1 \\ -1 & 2 \end{bmatrix} \end{aligned}$$

To compute Q:

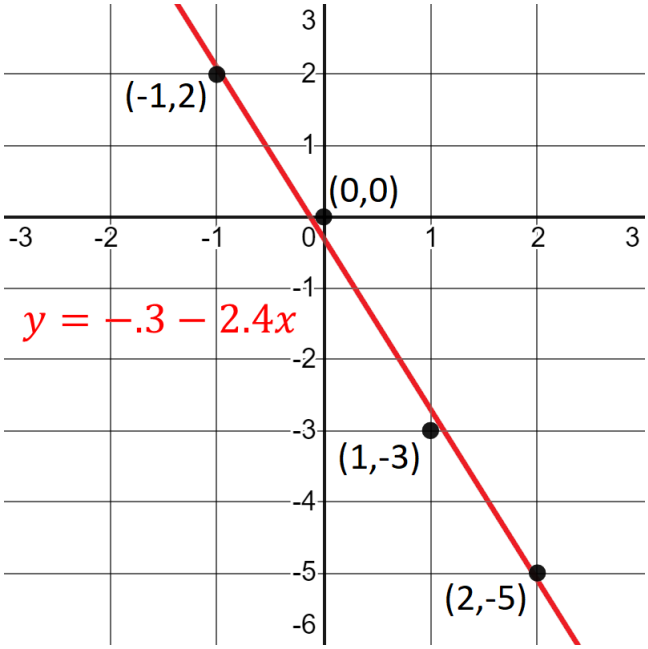
$$\begin{aligned} Q &= (A^TA)^{-1}A^T \\ &= \frac{1}{10} \begin{bmatrix} 3 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 0 & 1 & 2 \end{bmatrix} \\ &= \frac{1}{10} \begin{bmatrix} 3+1 & 3+0 & 3-1 & 3-2 \\ -1-2 & -1+0 & -1+2 & -1+4 \end{bmatrix} \\ &= \frac{1}{10} \begin{bmatrix} 4 & 3 & 2 & 1 \\ -3 & -1 & 1 & 3 \end{bmatrix} \end{aligned}$$

Finally, $\widehat{x} = Qb$ means

$$\widehat{x} = \begin{bmatrix} \widehat{C} \\ \widehat{D} \end{bmatrix} = \frac{1}{10} \begin{bmatrix} 4 & 3 & 2 & 1 \\ -3 & -1 & 1 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ -3 \\ -5 \end{bmatrix}$$

$$= \frac{1}{10}\begin{bmatrix} 8+0-6-5 \\ -6+0-3-15 \end{bmatrix}$$

$$= \frac{1}{10}\begin{bmatrix} -3 \\ -24 \end{bmatrix}$$

$$= \begin{bmatrix} -.3 \\ -2.4 \end{bmatrix}$$

We conclude $y = \widehat{C} + \widehat{D}t = -.3 - 2.4t$ is the line of best fit.